MMAT3560 200224

Simplex Method Explained

Basic Form

feasible

$$
\begin{array}{c|c|c}
 & \vec{y}_I & -1 \\
\hline
\vec{x}_I & A & \vec{b}^T \\
\hline
-1 & \vec{c} & -d
\end{array}
\quad
\begin{array}{l}
\\
= -\vec{y}_B \\
= f
\end{array}
$$

$$\vec{c} = \vec{x}_B, \qquad -d = g$$

$$A\vec{y}_I^{\,T} - \vec{b}^T = -\vec{y}_B \leq \vec{0}$$

$$\vec{x}_I A - \vec{c} = \vec{x}_B \geq \vec{0}$$

1. The variables at the bottom row ($\vec{x}_B$) and at the rightmost column ($\vec{y}_B$) are called basic varibles. The other variables at the leftmost column ($\vec{x}_I$) and at the top row ($\vec{y}_I$) are called non-basic variables or independent variables.

Initially $\vec{x} = \vec{x}$, $\vec{x} = \vec{x}$

Similarly $\vec{x}_I = \vec{x}$, $\vec{x}_B = \vec{x}_s$

$$\vec{y}_I = \vec{y}, \quad \vec{y}_B = \vec{y}_s$$

2. The solution obtained by setting $\vec{x}_I = \vec{y}_I = \vec{0}$ and $\vec{x}_B = -\vec{c}$, $\vec{y}_B = \vec{b}$ is called the <u>basic solution</u> of the basic form.
3. At the basic solution, we have
$$g(\vec{x}) = f(\vec{y}) = d$$
4. The basic solution may not give you feasible vectors.
5. The basic solution is feasible if and if $\vec{c} \le \vec{0}$ and $\vec{b} \ge \vec{0}$
6. The pivot operation changes the basic

form but the resulting basic form is equivalent to the original one.

$$
\begin{array}{c|cc|l}
 & y_l & y_j & \\
\hline
x_k & a^* & b & = -y_{n+k} \\
x_i & c & d & = -y_{n+i} \\
\hline
 & \shortparallel & \shortparallel & \\
 & x_{m+l} & x_{m+j} &
\end{array}
$$

$$
\begin{cases} a y_l + b y_j = -y_{n+k} \\ c y_l + d y_j = -y_{n+i} \end{cases}
$$

x_k

$$
\begin{cases} a x_k + c x_i = x_{m+l} \\ b x_k + d x_i = x_{m+j} \end{cases}
$$

$$
\Longleftrightarrow \begin{cases} y_{n+k} + b y_j = -a y_l \\ c y_l + d y_j = -y_{n+i} \end{cases}
$$

$$
\Longleftrightarrow \begin{cases} \frac{1}{a} y_{n+k} + \frac{b}{a} y_j = -y_l \\ c y_l + d y_j = -y_{n+i} \end{cases}
$$

$$\Leftrightarrow \begin{cases} \frac{1}{a} y_{n+k} + \frac{b}{a} y_j = -y_l \\ c\left(-\frac{1}{a} y_{n+k} - \frac{b}{a} y_j\right) + d y_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a} y_{n+k} + \frac{b}{a} y_j = -y_l \\ -\frac{c}{a} y_{n+k} + \left(d - \frac{bc}{a}\right) y_j = -y_{n+i} \end{cases}$$

$$\begin{array}{c|cc|c} & y_{n+k} & y_j & \\ \hline x_{m+l} & \frac{1}{a} & \frac{b}{a} & = -y_l \\ x_i & -\frac{c}{a} & d - \frac{bc}{a} & = -y_{n+i} \\ \hline & \| & \| & \\ & x_k & x_{m+j} & \end{array}$$

Example: $A = \begin{pmatrix} 1 & -1 & 3 \\ -3 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix}$

Sol. Add $k=3$ to every entry

$$\begin{pmatrix} 4 & 2 & 6 \\ 0 & 4 & 5 \\ 3 & 6 & 2 \end{pmatrix}$$

	y_1	y_2	y_3	-1		
x_1	4^*	2	6	1	$= -y_4$	$\frac{1}{4}$
x_2	0	4	5	1	$= -y_5$	
x_3	3	6	2	1	$= -y_6$	$\frac{1}{3}$
-1	1	1	1	0		
	$\parallel$	$\parallel$	$\parallel$			
	x_4	x_5	x_6			

$\longrightarrow$

	x_1	y_2	y_3	
y_1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{4}$
x_2	0	4	5	1
x_3	$-\frac{3}{4}$	$\frac{9}{2}^*$	$-\frac{5}{2}$	$\frac{1}{4}$
	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$

$\longrightarrow$

	x_1 (y_4)	x_3 (y_6)	y_3		
x_4 y_1	$\frac{1}{3}$	$-\frac{1}{9}$	$\frac{16}{9}$	$\frac{2}{9}$	
x_2	$\frac{2}{3}$	$-\frac{8}{9}$	$\frac{65}{9}$	$\frac{7}{9}$	$= -y_5$
y_2	$-\frac{1}{6}$	$\frac{2}{9}$	$-\frac{5}{9}$	$\frac{1}{18}$	
	$-\frac{1}{6}$	$-\frac{1}{9}$	$-\frac{2}{9}$	$-\frac{5}{18}$	
	$\parallel$ x_1	$\parallel$ x_3	$\parallel$ x_6		

$\vec{x} = (x_1, x_2, x_3) = (\frac{1}{6}, 0, \frac{1}{9})$ $\quad$ $\vec{x}_s = (x_4, x_5, x_6) = (0, 0, \frac{2}{9})$

$\vec{y} = (y_1, y_2, y_3) = (\frac{2}{9}, \frac{1}{18}, 0)$ $\quad$ $\vec{y}_s = (y_4, y_5, y_6) = (0, \frac{7}{9}, 0)$

$d = \frac{5}{18}$

maximin strategy: $\vec{p} = \frac{1}{d}\vec{x} = \frac{18}{5}(\frac{1}{6}, 0, \frac{1}{9}) = (\frac{3}{5}, 0, \frac{2}{5})$

minimax strategy: $\vec{q} = \frac{1}{d}\vec{y} = \frac{18}{5}(\frac{2}{9}, \frac{1}{18}, 0) = (\frac{4}{5}, \frac{1}{5}, 0)$

value of A: $v = \frac{1}{d} - k = \frac{18}{5} - 3 = \frac{3}{5}$

Check: $\vec{p}A = (\frac{3}{5}, 0, \frac{2}{5})\begin{pmatrix} 1 & -3 & 3 \\ -3 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix} = (\frac{3}{5}, \frac{3}{5}, \frac{7}{5})$

$$A\vec{q}^T = \begin{pmatrix} 1 & -3 & 3 \\ -3 & 1 & 2 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} 4/5 \\ 1/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -11/5 \\ 3/5 \end{pmatrix}$$

Minimax theorem

Thm. Given any $m \times n$ matrix A.

$\exists\ \vec{p} \in \mathcal{P},\ \vec{q} \in \mathcal{P},\ v \in \mathbb{R}$ s.t.

$\vec{p}A \geq v\vec{1}$ and $A\vec{q}^T \leq v\vec{1}^T$

Def. 1. row value of A:

$$v_r(A) = \max_{\vec{x} \in \mathcal{P}^m} \Big(\underbrace{\min_{\vec{y} \in \mathcal{P}^n} \vec{x} A \vec{y}^T}_{\text{depends on } \vec{x}} \Big)$$

maximin value

2. column value of A:

$$v_c(A) = \min_{\vec{y} \in \mathcal{P}^n} \Big(\underbrace{\max_{\vec{x} \in \mathcal{P}^m} \vec{x} A \vec{y}^T}_{\text{depends on } \vec{y}} \Big)$$

minimax value

Minimax thm: $v_r(A) = v_c(A)$ for any $m \times n$ matrix A.

Consider $f(x,y) = |x-y|$, $x, y \in [0,1]$

$\max \min |x-y| = 0$

$x \to [0,1] \quad y \in [0,1]$ ... 0

$\nwarrow$ take $y = x$

$$\min_{y\in[0,1]} \max_{x\in[0,1]} |x-y| = \frac{1}{2}$$

Thm. $V_r(A) \le V_c(A)$

Proof. $V_r(A) = \min_{\vec{y}\in P^n} \vec{p}A\vec{y}^T$ $\qquad$ $\vec{p}$ is a maximin strategy

$\qquad \min_{\vec{y}\in P^n} \vec{p}A\vec{y}^T = V_r(A)$

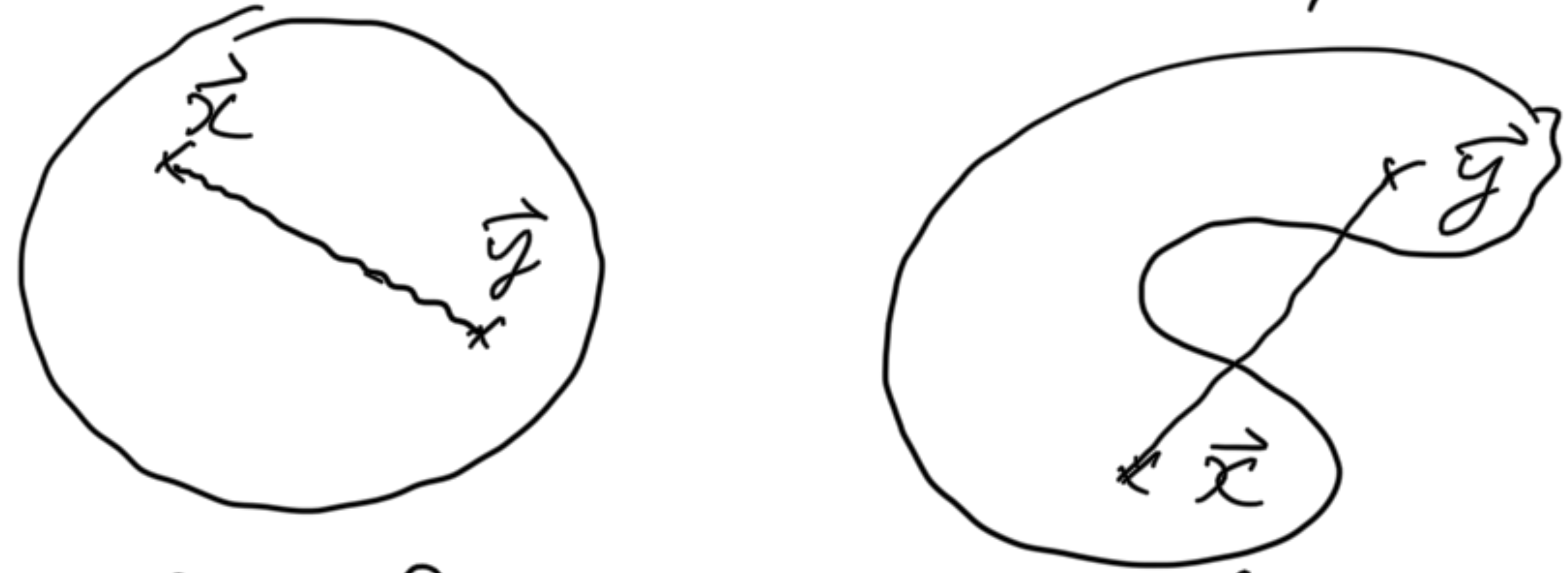
$$\le \vec{p}A\vec{q}^T$$

$$\le \max_{\vec{x}\in P^m} \vec{x}A\vec{q}^T$$ $\qquad$ $\vec{q}$ is a minimax strategy

$$= V_c(A) \qquad \square$$

Def. A subset $C \subset \mathbb{R}^n$ is said to be convex if

$\underline{\lambda\vec{x} + (1-\lambda)\vec{y} \in C}$ for any $\vec{x}, \vec{y} \in C$, $0 \le \lambda \le 1$



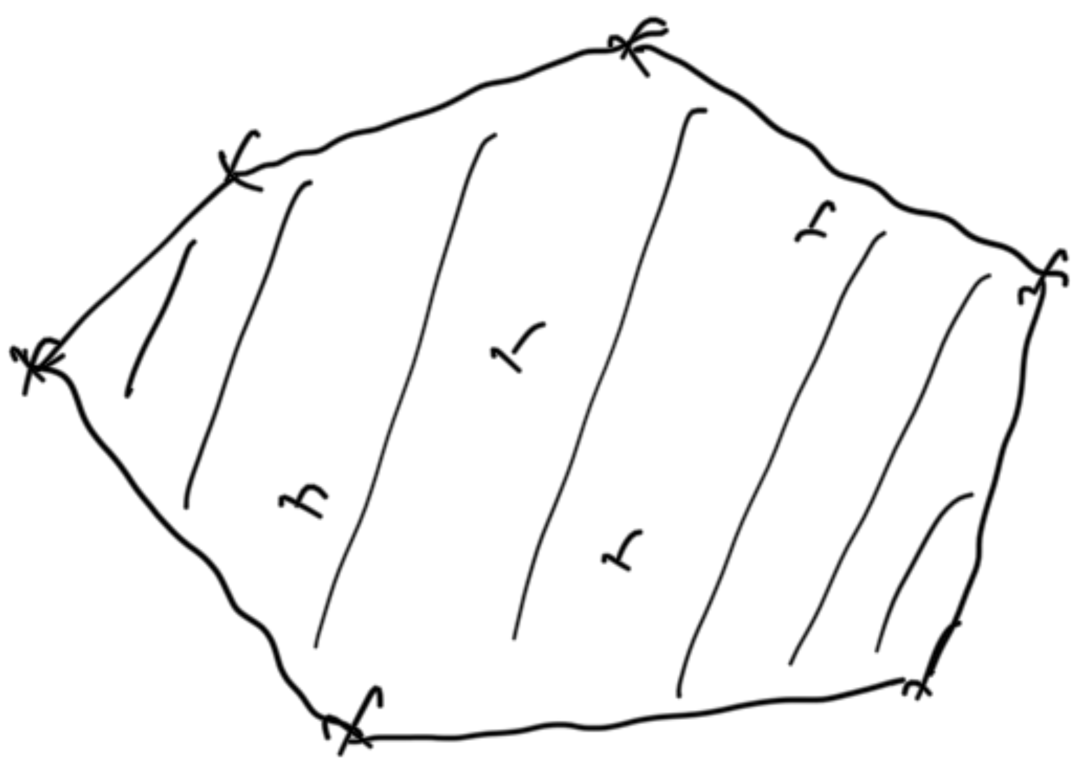
convex not convex

The convex hull of $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\} \subset \mathbb{R}^n$ is

$$\mathrm{Conv}(\{\vec{x}_1, \cdots, \vec{x}_k\}) = \{\vec{x} \in \mathbb{R}^n : \vec{x} = \lambda_1\vec{x}_1 + \lambda_2\vec{x}_2 + \cdots + \lambda_k\vec{x}_k$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \geq 0$

$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1\}$

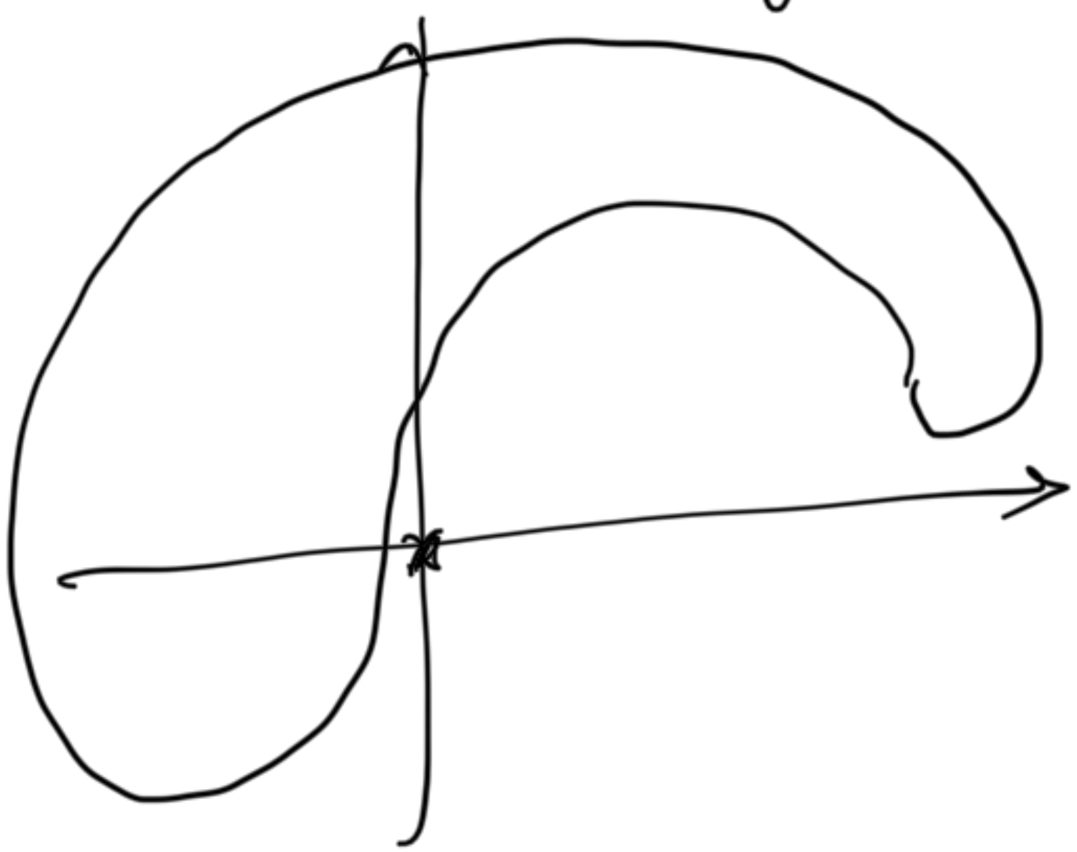


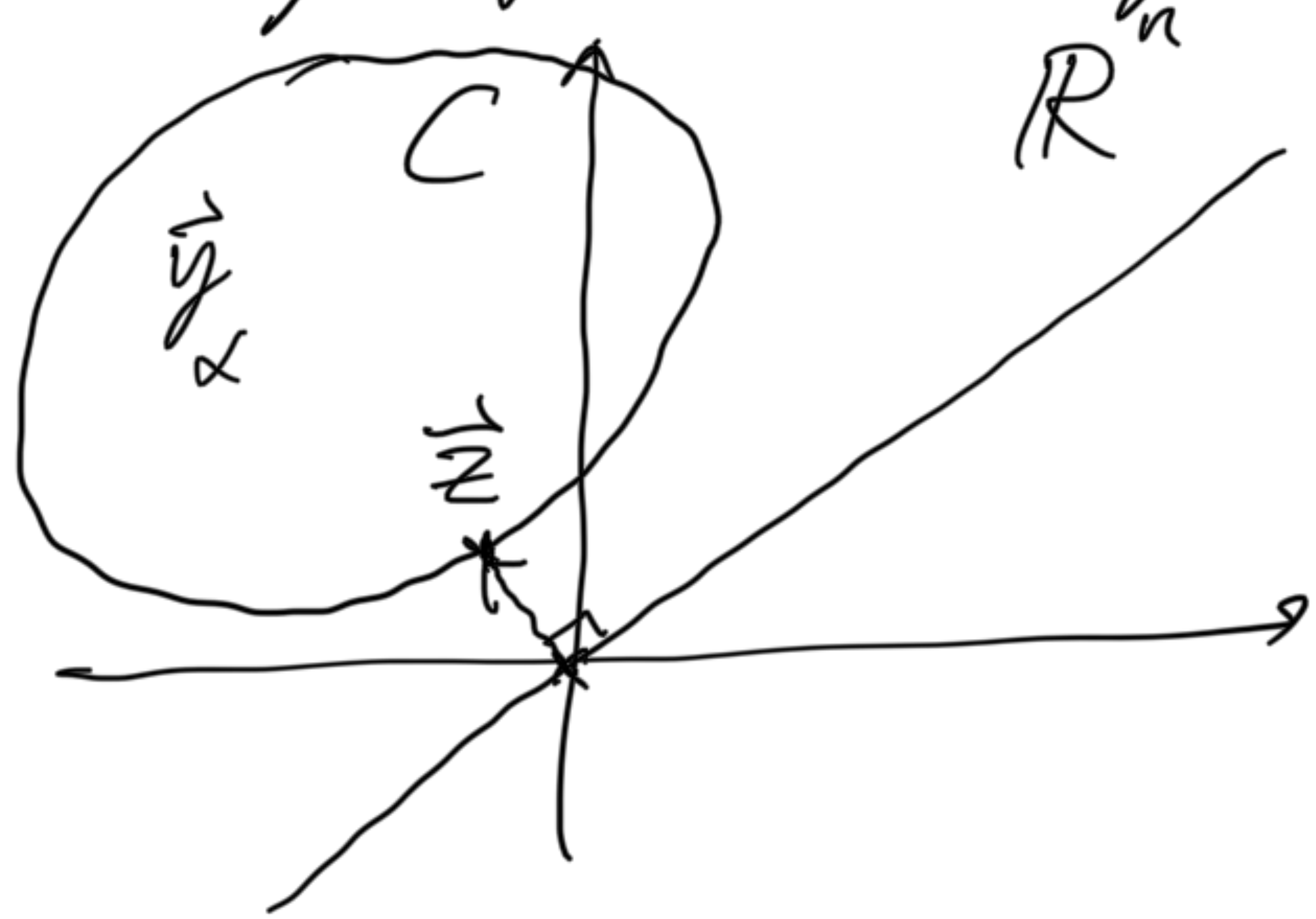
The convex hull is the smallest convex set containing $\vec{x}_1, \cdots, \vec{x}_k$.

Lemma (Special case of hyperplane separation theorem)

Let $C \subset \mathbb{R}^n$ be a <u>closed</u> convex set and $\vec{0} \notin C$. Then $\exists\, \vec{z} \in C$ s.t.

$\langle \vec{z}, \vec{y} \rangle > 0$ for any $\vec{y} \in C$ $\quad \langle \vec{z}, \vec{y} \rangle = \vec{z}\vec{y}^T$





Proof. Take $\vec{z} \in C$ s.t.

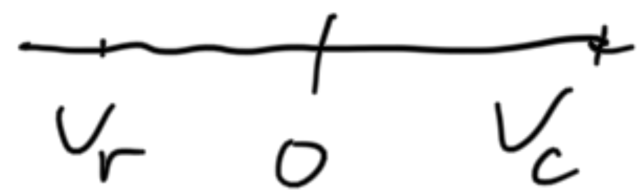
$$|\vec{z}| = \min_{\vec{y} \in C} |\vec{y}|$$

Then $\langle \vec{z}, \vec{y} \rangle > 0 \quad \forall \vec{y} \in C$

Thm. Let A be an $m \times n$ matrix.

Then either

1. $\exists\, \vec{x} \in P^m$ s.t. $\vec{x}A > \vec{0}$, in this case $V_r(A) > 0$, or
2. $\exists\, \vec{y} \in P^n$ s.t $A\vec{y}^T \leq \vec{0}$, in this case $V_c(A) \leq 0$.



cannot happen

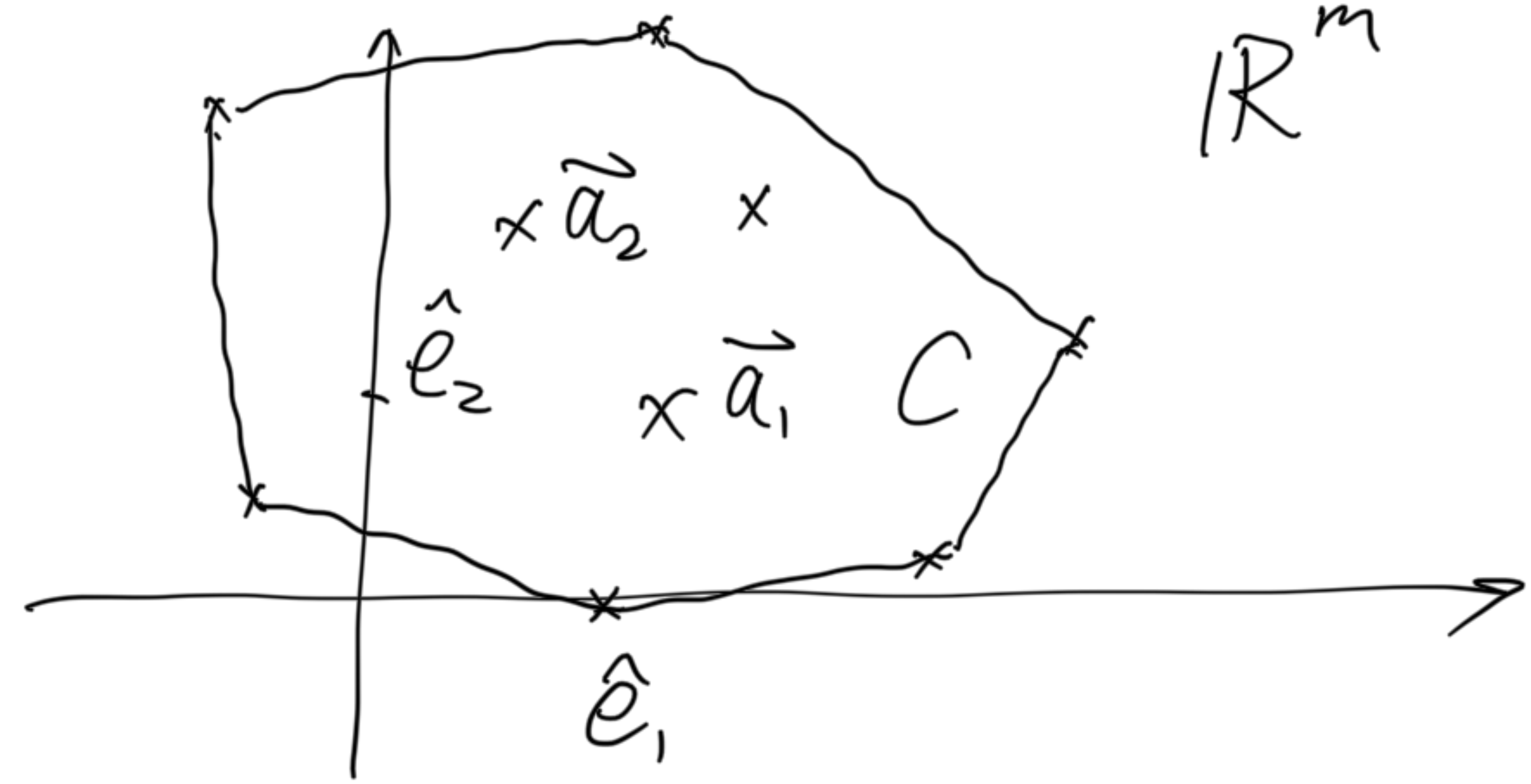
✓

Proof. Let $C = \mathrm{Conv}(\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n, \hat{e}_1, \hat{e}_2, \cdots \hat{e}_m\}) \subset \mathbb{R}^m$

where $\vec{a}_1^T, \cdots, \vec{a}_n^T$ are column vectors of A

$$A = [\vec{a}_1^T \vec{a}_2^T \cdots \vec{a}_n^T]$$

$\hat{e}_1, \cdots, \hat{e}_m$ vectors in the standard basis.



C is a closed convex set.

① Suppose $\vec{0} \notin C$.

Then by the lemma, $\exists \vec{z} \in C$ s.t.

$$\langle \vec{z}, \vec{y} \rangle > 0 \quad \forall \vec{y} \in C$$

In particular, Take $\vec{y} = \hat{e}_1, \dots, \hat{e}_m$.

$$\langle \vec{z}, \hat{e}_i \rangle = z_i > 0 \qquad \vec{z} = (z_1, \dots, z_m)$$

Take $\vec{x} = \frac{1}{z_1 + \dots + z_m} \vec{z} \in P^m$

$$\vec{x}A = \vec{x}[a_1\ a_2\ \cdots\ a_n]$$

$$= [\vec{x}\vec{a}_1^T, \vec{x}\vec{a}_2^T, \cdots, \vec{x}\vec{a}_n^T]$$

$$> \vec{0} \quad (\langle \vec{x}, \vec{a}_i \rangle = \vec{x}\vec{a}_i^T > 0)$$

$\vec{a}_i \in C$

② Suppose $\vec{0} \in C$.

Then $\exists\ \lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}, \cdots, \lambda_{n+m} \geq 0$

with $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$

s.t. $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \cdots + \lambda_n\vec{a}_n + \lambda_{n+1}\hat{e}_1 + \cdots + \lambda_{m+n}\hat{e}_m = \vec{0}$

$$\Rightarrow A\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} = -\begin{pmatrix}\lambda_{n+1}\\ \vdots \\ \lambda_{n+m}\end{pmatrix} = -\lambda_{n+1}\hat{e}_1 - \lambda_{n+2}\hat{e}_2 + \cdots + (-\lambda_{n+m}\hat{e}_m)$$

Take $\vec{y} = \dfrac{1}{\lambda_1 + \cdots + \lambda_n}(\lambda_1, \cdots, \lambda_n) \in P^n$

At least one of $\lambda_1, \cdots, \lambda_n$ is nonzero

because $\hat{e}_1, \cdots, \hat{e}_m$ are linearly independent.

$$\Lambda \vec{y}^{T} = \frac{-1}{\lambda_1 + \cdots + \lambda_n} \begin{pmatrix} \lambda_{n+1} \\ \vdots \\ \lambda_{n+m} \end{pmatrix}$$